If you have any question, feel free to come to my office (N7.905) or to send me an email (emilie.despontin@ulb.be)

Problem set 1 : Quasi-circular inspirals

I. Introduction & Motivations

Why is it of interest ?

Today's exercises will help you to understand the basic mechanisms behind GW generation, propagation and detection. We will start from first principles : how moving masses disturb spacetime, and build up to realistic astrophysical sources like binary inspirals.

Historical context

- 1916 : Einstein's prediction

  ↳ Einstein's GR predicts that accelerating masses should generate GW. However, himself doubted they would ever be detected because of their incredibly weak effect on matter.

- 1960 - 1980's : indirect evidence

  ↳ The first evidence of GW came in 1974 when Hulse and Taylor observed a binary pulsar losing energy exactly as predicted by GR

  Their discovery confirmed the existence of GW → '93 Nobel

- 2015 : first direct detection GW150914

  ↳ LIGO detectors observed GW from the merger of 2 BHs $(36 M_\odot - 29 M_\odot)$

- Today

  ↳ LIGO, Virgo + LISA

Overview of today's session

- Quadrupolar radiation

  ↳ Why do GW emerge from mass-energy motion ?
  Fundamental eq governing GW generation
  How stress-energy conservation leads to quadrupolar emission

- Quasi-circular inspiral

  ↳ Understanding binary orbits and their evolution
  Deriving the GW frequency and waveform
  Relating theory to real astrophysics detection (eg GW150914)

- Applications to real observations

  ↳ How we connect mathematical prediction to real data
  Calculating properties of detected GW
  Insights from actual GW events and how they confirm our models.

Some key concepts

Metric perturbations and the TT-gauge

In GR, spacetime is described by a metric $g_{\mu\nu}$. We consider small perturbation around flat spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$

We will choose the TT-gauge to simplify calculations, which we can choose only outside the source. We have only two physical dof for GWs.

Quadrupole formula

The leading order GW emission is quadrupolar (not monopolar or dipolar, due to mass-energy conservation)

The strain observed at a distance r is given by

$$h_{ij}^{TT} = \frac{2G}{rc^4} \Lambda_{ij,kl} \ddot{M}_{kl}$$

↳ mass quadrupole moment of the source

Binary inspirals and Chirp mass

A binary inspiral loses energy through GW emission, causing the orbit to shrink. The key parameter controlling this evolution is the chirp mass

$$M_c = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}.$$

The GW frequency increases over time (="chirp") following

$$f_{GW} = \frac{1}{\pi}\left(\frac{5}{256}\cdot\frac{1}{\tau}\right)^{3/8}\left(\frac{G M_c}{c^3}\right)^{-5/8}$$

where τ = time until coalescence.

II. QUADRUPOLAR GRAVITATIONAL RADIATION

Problem 1.1 (Conservation laws for stress-energy moments)

a) In linearized gravity, the conservation of the stress-energy tensor is expressed as $\partial_\mu T^{\mu\nu} = 0$.
We define the energy density moments as

$$M^{i_1 \dots i_\ell} = \frac{1}{c^2} \int d^3x \, T^{00}(t, \vec{x})\, x^{i_1} \dots x^{i_\ell}$$

and the linear momentum moments as

$$P^{j, i_1 \dots i_\ell} = \frac{1}{c} \int d^3x \, T^{0j}(t, \vec{x})\, x^{i_1} \dots x^{i_\ell}.$$

Show that in a volume V enclosing the source entirely, we have

$$\dot{M}^{i_1 \dots i_\ell} = \ell . P^{(i_1, i_2 \dots i_\ell)}$$

$$\dot{P}^{j, i_1 \dots i_\ell} = \ell \, S^{j(i_1, i_2 \dots i_\ell)}.$$

with

$$P^{(i_1 \dots i_\ell)} = \frac{1}{\ell!} \left((\ell-1)!\, P^{i_1 \dots i_\ell} + (\ell-1)!\, P^{i_2 i_1 \dots i_\ell} + \dots \right).$$

We have

- $$c . \dot{M}^{i_1 \dots i_\ell} = \frac{1}{c} \partial_t \int d^3x \, T^{00}(t, \vec{x})\, x^{i_1} \dots x^{i_\ell}$$
$$= \int_V d^3x \, \partial_0 T^{00}(t, \vec{x})\, x^{i_1} \dots x^{i_\ell}$$ $\partial_\mu T^{\mu\nu} = 0$
$$= \int_V d^3x \left[-\partial_j T^{0j} . x^{i_1} \dots x^{i_\ell} \right]$$ by parts (T cancels at the bdy of V)
$$= \int_V d^3x \, T^{0j} \partial_j (x^{i_1} \dots x^{i_\ell})$$
$$= \int_V d^3x \, T^{0j} \left[\delta^{i_1}_j x^{i_2} \dots x^{i_\ell} + x^{i_1} \delta^{i_2}_j \dots x^{i_\ell} + \dots \right]$$
$$= c \left[P^{i_1, i_2 \dots i_\ell} + P^{i_2, i_1 \dots i_\ell} + \dots \right]$$
$$= c \ell P^{(i_1, i_2 \dots i_\ell)}$$

- $$\dot{P}^{j, i_1 \dots i_\ell} = \int_V d^3x . \partial_0 T^{0j}(t, \vec{x})\, x^{i_1} \dots x^{i_\ell}$$
$$= -\int_V d^3x . \partial_i T^{ij} . x^{i_1} \dots x^{i_\ell}$$
$$= \int_V d^3x . T^{ij} \left[\delta^{i_1}_i x^{i_2} \dots x^{i_\ell} + \dots \right]$$
$$= \ell S^{j(i_1, i_2 \dots i_\ell)}$$

b) Use your results to reexpress h_{ij}^{TT} in terms of the moments of $T^{\alpha\mu}$ up to subleading in $1/c$.

We have

$$h_{ij}^{TT} = \frac{1}{r} \cdot \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{n}) \left[S^{kl} + \mathcal{O}(1/c) \right]$$

and

$$\begin{cases} S^{kl} = \dot{P}^{kl} \\ \ddot{M}^{kl} = \dot{P}^{kl} + \dot{P}^{lk} \end{cases} \Rightarrow S^{kl} = \frac{1}{2} \ddot{M}^{kl}$$

$$\Rightarrow h_{ij}^{TT} = \frac{1}{r} \cdot \frac{4G}{c^4} \Lambda_{ij,kl} \left[\frac{1}{2} \underbrace{\ddot{M}^{kl} \Big|_{t-r/c}}_{\text{quadrupolar moment}} + \mathcal{O}(1/c) \right]$$

$\hookrightarrow$ $\ddot{M}^{ij} = \dot{P}^{ij} + \dot{P}^{ji} = S^{ij} + S^{ji} = 2S^{ij}$

Problem 1.2 (Angular distribution of quadrupolar radiation)

a) Consider that the radiation is aligned with an axis of the detector frame $\{x, y, z\}$, say $\hat{n} = \hat{z}$.
Show that the two physical polarisations are given by

$$h_+ = \frac{1}{r}\cdot\frac{G}{c^4}\left(\ddot{M}_{11} - \ddot{M}_{22}\right) \quad ; \quad h_\times = \frac{2}{r}\cdot\frac{G}{c^4}\ddot{M}_{12}$$

at $t - r/c$.

Having $\hat{n} = \hat{z}$ implies

$P^{ij} = \delta^{ij} - n^i n^j$

$$P^{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}.$$

We have to compute

$$h_{ij}^{TT} \simeq \frac{1}{r}\cdot\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{n})\,\ddot{M}^{kl}\Big|_{t-r/c}$$

We know that

$$\Lambda_{ij,kl}A^{kl} = \left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right)A^{kl}$$
$$= (PAP)_{ij} - \frac{1}{2}P_{ij}\,\mathrm{tr}(PA)$$

and

$$\begin{cases} \mathrm{tr}(PA) = A_{11} + A_{22} \\ PAP = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{cases}$$

$$\Rightarrow \Lambda_{ij,kl}A^{kl} = \begin{pmatrix} \frac{1}{2}(A_{11} - A_{22}) & A_{12} & 0 \\ A_{21} & \frac{1}{2}(A_{22} - A_{11}) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow h_{ij}^{TT} = \frac{1}{r}\cdot\frac{2G}{c^4}\begin{pmatrix} \frac{1}{2}(\ddot{M}_{11} - \ddot{M}_{22}) & \ddot{M}_{12} & 0 \\ \ddot{M}_{21} & \frac{1}{2}(\ddot{M}_{22} - \ddot{M}_{11}) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} h_+ = \frac{1}{r}\,\frac{G}{c^4}\left(\ddot{M}_{11} - \ddot{M}_{22}\right) \\ h_\times = \frac{1}{r}\cdot\frac{2G}{c^4}\ddot{M}_{12} \end{cases}$$

cf (1.46) lecture notes

b) For $\hat{m}$ not aligned with $\hat{z}$, consider a new frame $\{u, v, m\}$ st $\begin{cases} u \times v = m \\ u \text{ lies in the } (x,y)\text{-plane} \end{cases}$

In that frame, the wave propagates along m. Write its expression

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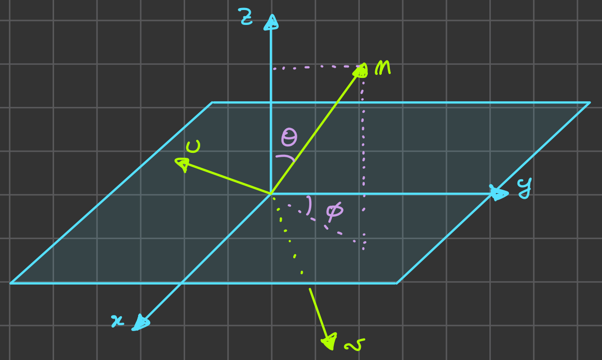
We have the following :



- in the $\{x, y, z\}$-basis, we have

$$m_i = \{\sin\theta \sin\phi,\ \sin\theta\cos\phi,\ \cos\theta\} \quad \text{(spherical coord.)}$$

- in the $\{u, v, m\}$-basis, we have

$$m_i' = \{0, 0, 1\}$$

We can relate those components by a rotation matrix R st

$$m_i = R_{ij}\, m_j'$$

where

$$R = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

c) Show that in the original frame, we have

$$\begin{cases} h_+ = \frac{1}{r}\cdot\frac{G}{c^4}\Big[\ddot{M}_{11}(\cos^2\phi - \sin^2\phi\cos^2\theta) + \ddot{M}_{22}(\sin^2\phi - \cos^2\phi\cos^2\theta) - \ddot{M}_{33}\sin^2\theta \\ \qquad\qquad - \ddot{M}_{12}\sin 2\phi\,.(1+\cos^2\theta) + \ddot{M}_{13}\sin\phi\sin 2\theta + \ddot{M}_{23}\cos\phi\sin 2\theta\Big] \\ h_\times = \frac{1}{r}\cdot\frac{G}{c^4}\Big[(\ddot{M}_{11} - \ddot{M}_{22})\sin 2\phi\cos\theta + 2\big(\ddot{M}_{12}\cos 2\phi\cos\theta - \ddot{M}_{13}\cos\phi\sin\theta + \ddot{M}_{23}\sin\phi\sin\theta\big)\Big] \end{cases}$$

Similarly as the m-vector, a tensor M with 2 indices has components M_{ij} in the $\{x, y, z\}$-frame and M_{ij}' in the $\{u, v, m\}$-frame st they're related by

$$M_{ij} = R_{ik} R_{jl} M_{kl}'$$

$$\Leftrightarrow M_{ij}' = (R^{\sim} M R)_{ij}$$

Then, we insert R to obtain M_{ij}' and replacing

$$\begin{cases} h_+ = \frac{1}{r}\cdot\frac{G}{c^4}\left(\ddot{M}_{11}' - \ddot{M}_{22}'\right) = \dots \\ h_\times = \frac{2}{r}\cdot\frac{G}{c^4}\,\ddot{M}_{12}' = \dots \end{cases}$$

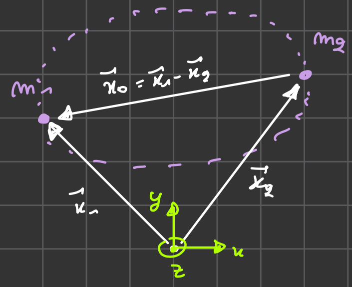
look p.127 Magg. for comments

↳ this allows to compute the angular distribution of the quadrupole radiation, given M_{ij}.

III. QUASI-CIRCULAR INSPIRAL

We now consider a system of two point masses m_1 and m_2. Their motion happens in the (x,y)-plane. Their relative coord. $\vec{x}_0 = \vec{x}_1 - \vec{x}_2$ follows a circular motion

$$\begin{cases} x_0(t) = R.\cos\left(\omega_s t + \frac{\pi}{2}\right) \\ y_0(t) = R.\sin\left(\omega_s t + \frac{\pi}{2}\right) \\ z_0(t) = 0 \end{cases}$$



with R the orbital radius.

Problem 2.1

Show that the quadrupolar mass moment is given by

$$M^{ij} = m_1 x_1^i x_1^j + m_2 x_2^i x_2^j .$$

For a point-like particle moving on a given trajectory $x_0(t)$, in flat spacetime, the energy-momentum tensor is

$$T^{\mu\nu}(t, x) = \frac{p^\mu p^\nu}{\gamma m} \delta^{(3)}(\vec{x} - \vec{x}_0(t))$$

w/ $\gamma = (1 - v^2/c^2)^{-1/2}$ and $p^\mu = (E/c, \vec{p})$. For a set of point-particles moving under their mutual influence on trajectories $x_A^\mu(t)$,

$$T^{\mu\nu}_{tot}(t, \vec{x}) = \sum_A \frac{p_A^\mu p_A^\nu}{\gamma_A m_A} \delta^{(3)}(\vec{x} - \vec{x}_A(t))$$

$$\Rightarrow T^{00}_{tot} = \sum_A \gamma^A m_A c^2 \delta^{(3)}(\vec{x} - \vec{x}_A(t)).$$

In our case, we have

$$T^{00} = m_1 c^2 \delta^{(3)}(\vec{x} - \vec{x}_1) + m_2 c^2 \delta^{(3)}(\vec{x} - \vec{x}_2)$$

such that

$$M^{ij} = \int d^3x \left[m_1 \delta^{(3)}(\vec{x} - \vec{x}_1) + m_2 \delta^{(3)}(\vec{x} - \vec{x}_2) \right] x^i x^j$$

$$= m_1 x_1^i x_1^j + m_2 x_2^i x_2^j$$

$$\vdots$$

$$= \mu x_0^i x_0^j$$

→ quadrupole moment ≡ to the one of a particle of mass μ described by the coord. x_0.

in the CM frame, with the reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

If we opted for a non-vanishing x_{CM},

$$M^{ij} = \underbrace{m x^i_{CM} x^j_{CM}}_{c^{te}} + \underbrace{\mu x^i_{CM} x_0^j + \mu x^j_{CM} x_0^i}_{\text{contribute}} + \mu x_0^i x_0^j$$

⇒ doesn't contribute to the production of grav. radiation

see p138

Problem 2.2

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a) Compute the mass moments. Show that

$$\begin{cases} h_+ = \frac{4G}{rc^4}\mu R^2 \omega_s^2 \left(\frac{1+\cos^2\theta}{2}\right)\cos(2\omega_s t_{ret} + 2\phi) \\ h_\times = \frac{4G}{rc^4}\mu R^2\omega_s^2 \cos\theta \sin(2\omega_s t_{ret} + 2\phi) \end{cases}$$

Notice that $\omega_{GW} = 2\omega_s$.

We have that

$$\begin{cases} M_{11} = \mu R^2 \cos^2\left(\omega_s t + \frac{\pi}{2}\right) \\ \quad = \mu R^2 \frac{1+\cos(2\omega_s t + \pi/2)}{2} \\ \quad = \mu R^2 \frac{1-\cos(2\omega_s t)}{2} \\ M_{12} = -\frac{\mu R^2 \sin(2\omega_s t)}{2} \\ M_{22} = \mu R^2 \frac{1+\cos(2\omega_s t)}{2} \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{M}_{11} = 2\mu R^2 \omega_s^2 \cos(2\omega_s t) \\ \ddot{M}_{12} = -\ddot{M}_{21} \\ \ddot{M}_{22} = 2\mu R^2\omega_s^2 \sin(2\omega_s t) \end{cases}$$

$\to$ TT-gauge $\Rightarrow$ traceless :")

By replacing it in (12), we get (17).

b) Compute the angular distribution of the radiated power

$$\left.\frac{dP}{d\Omega}\right|_{quad.} = \frac{r^2c^3}{32\pi G}\langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\rangle$$

Notice that there is no angle for which no power is received. Integrate this distribution to find the total radiated power

$$P_{quad.} = \frac{32G\mu^2R^4\omega_s^6}{5c^5}.$$

We have

$$\left.\frac{dP}{d\Omega}\right|_{quad.} = \frac{r^2c^3}{32\pi G}\langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\rangle = \frac{r^2c^3}{32\pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle$$

Moreover,

$$\dot{h}_+ = -\frac{4G}{rc^4}\mu R^2\omega_s^2\left(\frac{1+\cos^2\theta}{2}\right)\sin(2\omega_s t)2\omega_s$$

such that

$$\begin{cases} \langle \dot{h}_+^2 \rangle = 64 \dfrac{G^2\mu^2 R^4 \omega_s^6}{r^2 c^8} \underbrace{\langle \sin^2(2\omega_s t)\rangle}_{= 1/2 \ \circledast} \left(\dfrac{1+\cos^2\theta}{2}\right)^2 \\ \langle \dot{h}_\times^2 \rangle = 32 \dfrac{G^2\mu^2 R^4 \omega_s^6}{r^2 c^8} \cos^2\theta \end{cases}$$

$\circledast$ $\langle f(t) \rangle = \frac{1}{T} \int_0^T f(t)\,dt$

w/ $T = \frac{2\pi}{2\omega_s} = \frac{\pi}{\omega_s}$; $\int \frac{1-\cos}{2}$

and $\int_0^T \cos(4\omega_s t) = 0$ bc periodic

$$\Rightarrow \left.\frac{dP}{d\Omega}\right|_{quad.} = 2 \frac{G\mu^2 R^4 \omega_s^6}{\pi c^5} g(\theta)$$

with $g(\theta) = \left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta > 0$. $\Rightarrow \nexists\, \theta$ st $g(\theta)$ vanishes

To find the total radiated power, we integrate st

$$\begin{aligned} \int d\Omega . g(\theta) &= \int \sin\theta\, d\theta\, d\phi . g(\theta) \\ &= 2\pi \int_{-1}^{1} dx . g(x) \\ &= \frac{2\pi}{4} \int_{-1}^{1} (1 + 6x^2 + x^4) \\ &= \frac{16\pi}{5} \end{aligned}$$

$\big)$ $x = \cos\theta$; $dx = -\sin\theta\, d\theta$

$$\Rightarrow \quad P_{quad.} = \frac{32 G\mu^2 R^4 \omega_s^6}{5c^5} = \frac{1}{10} \frac{G\mu^2 R^4 \omega_{GW}^6}{c^5} \qquad \text{w/ } \omega_{GW} = 2\omega_s$$

As energy is radiated, the orbital energy E_{orb} decreases $\Rightarrow$ R must also decrease:

$v = \omega_s R$

$$\begin{aligned} E_{orb.} &= E_k + E_p = \frac{1}{2} \underbrace{\mu v^2}_{\circledast} - G \frac{m_1 m_2}{R} \\ &= -\frac{G m_1 m_2}{2R} \end{aligned}$$

$v = \omega_s R$

Kepler: $\omega_s^2 = G \frac{m}{R^3}$

$\circledast$ $\frac{1}{2} \cdot \frac{m_1 m_2}{m_1 + m_2} \cdot \frac{G(m_1 + m_2)}{R^{\cancel{3}}} \cancel{R^2}$

Puissance : $P = -\frac{dE}{dt}$. (radiated power = lost energy)

Problem 2.3

Show that as long as $\dot{\omega}_s \ll \omega_s^2$, the radial velocity is negligible compared to the tangential one.

We have

$$\dot{R} = \frac{d}{dt}\left(\left(\frac{Gm}{\omega_s^2}\right)^{1/3}\right) = -\frac{2}{3} \omega_s R \frac{\dot{\omega}_s}{\omega_s^2}$$

$$\Rightarrow \dot{R} \ll \omega_s R \quad \text{iff} \quad \dot{\omega}_s \ll \omega_s^2$$

We introduce the chirp mass

$$\mathcal{M}_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

At the lowest order, frequency, polarisation... only depend on $\mathcal{M}_c$.

Problem 2.4

Show (23) and (24).

Enjoy some nice computations ☺

We can also express the radiated power in terms of $\mathcal{M}_c$:

$$P = \frac{32 c^5}{5G} \left(\frac{G \mathcal{M}_c \, \omega_{GW}}{2 c^3} \right)^{10/3}$$

Problem 2.5

Show that p183

$$\dot{f}_{GW} = C f_{GW}^{11/3}$$

with C a constant depending on the parameters of the system. Solve this eq. and discuss the regularity of $f_{GW}(t)$.

We have

$$E_{orb} = -\left(\frac{G^2 \mathcal{M}_c^5 \, \omega_{GW}^2}{32} \right)^{1/3}.$$

Indeed,

$$\omega_s^2 = \frac{Gm}{R^3} = \frac{1}{4} \omega_{GW}^2 \quad \text{and} \quad E_{orb} = -G \frac{m_1 m_2}{2R}.$$

Thus, we have

$$-\frac{dE_{orb}}{dt} = \frac{1}{3} \left(\frac{G^2 \mathcal{M}_c^5}{32} \right)^{1/3} \omega_{GW}^{-2/3} \, 2 \omega_{GW} \dot{\omega}_{GW} \overset{!}{=} P = \frac{32 c^5}{5G} \left(\frac{G \mathcal{M}_c \, \omega_{GW}}{2 c^3} \right)^{10/3}$$

$$\Leftrightarrow \dot{\omega}_{GW} = \frac{12}{5} 2^{1/3} \left(\frac{G \mathcal{M}_c}{c^3} \right)^{5/3} \omega_{GW}^{11/3}$$

$$f_{GW} = \frac{\omega_{GW}}{2\pi} \qquad \Leftrightarrow \dot{f}_{GW} = \frac{96}{5} \pi^{8/3} \left(\frac{G \mathcal{M}_c}{c^3} \right)^{5/3} f_{GW}^{11/3}$$

Solving this eq. diff, we find

$$f_{GW} = \frac{1}{\pi} \left(\frac{5}{256} \cdot \frac{1}{\tau} \right)^{3/8} \left(\frac{G \mathcal{M}_c}{c^3} \right)^{-5/8} \qquad (24)$$

with $\tau = t_{coal} - t$ and $t_{coal} \equiv$ time for which f_{GW} diverges. The divergence is cut-off by the fact that, when their separation becomes smaller than a critical distance, the 2 stars merge. The divergence is due to the point-particle approx. $\tau \equiv$ fusion time. Equivalently, we can write

$\mathcal{M}_c$ of 2 bodies of $1{,}4 M_\odot$ each

$$\tau \simeq 2{,}18\,s \left(\frac{1{,}21 M_\odot}{\mathcal{M}_c} \right)^{5/3} \left(\frac{100\,\text{Hz}}{f_{GW}} \right)^{8/3}$$

$M_\odot \simeq 10^{32}$

p185 for more interpret.

Problem 2.6

a) Using eq. 27, discuss the evolution of the orbital radius wrt time and the regime of validity of the quasicircular approximation.

We already computed

$$\dot R = -\frac{2}{3} R \frac{\dot\omega_s}{\omega_s}$$

We have

$$\dot\omega_{GW} = 2\pi \dot f_{GW} = 2\left(\frac{G\mathcal{M}_c}{c^3}\right)^{-5/8}\left(\frac{5}{256}\right)^{3/8}\frac{d}{dt}\left(\frac{1}{\tau^{3/8}}\right)$$

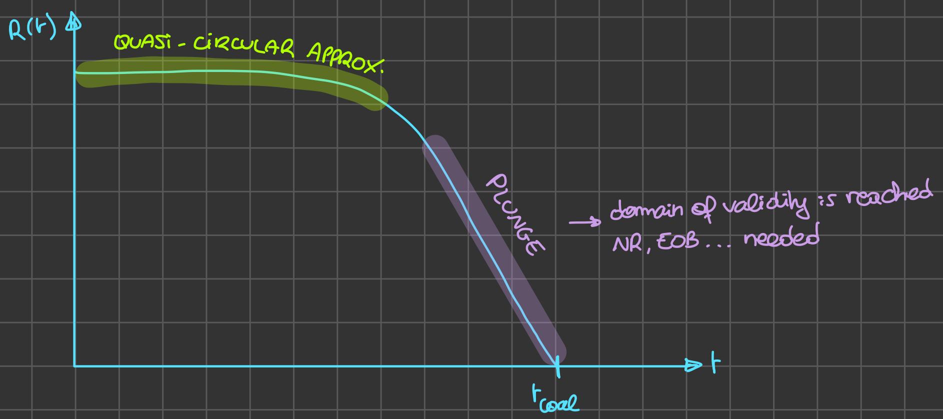
$$\frac{d}{dt}\left((t_{coal}-t)^{-3/8}\right) = \frac{3}{8}(t_{coal}-t)^{-11/8}(-1)$$

$$\vdots$$

$$= \frac{3}{8}\,\omega_{GW}\,\tau^{-1}$$

$$\Rightarrow \frac{\dot R}{R} = \frac{-1}{4\tau} \quad \Leftrightarrow \quad R(\tau) = R_0\left(\frac{t_{coal}-t}{t_{coal}-t_0}\right)^{1/4}$$

If we plot it, we have



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b) To compute the waveform associated to the quasi-circular inspiral, all appearances of $\omega_{GW}t$ have to be replaced with

$$\Phi(t) = \int_{t_0}^{t} \omega_{GW}(t')\,dt'$$

Compute $\Phi(t)$ and sketch the waveform associated to the inspiral.

We want to understand the signal received from Earth. We know that

$$\begin{cases} h_+ = \dots (\omega_{GW} t) \\ h_\times = \dots (\omega_{GW} t) \end{cases}$$

but we need to take into account that ω_{GW} is not constant. This is why we introduce

$$\Phi(t) = \int_{t_0}^{t} dt' \cdot \omega_{GW}(t')$$

$$= \int_{t_0}^{t} dt' \, 2 \left(\frac{5}{256}\cdot\frac{1}{\tau}\right)^{3/8} \left(\frac{G\mathcal{M}_c}{c^3}\right)^{-5/8}$$

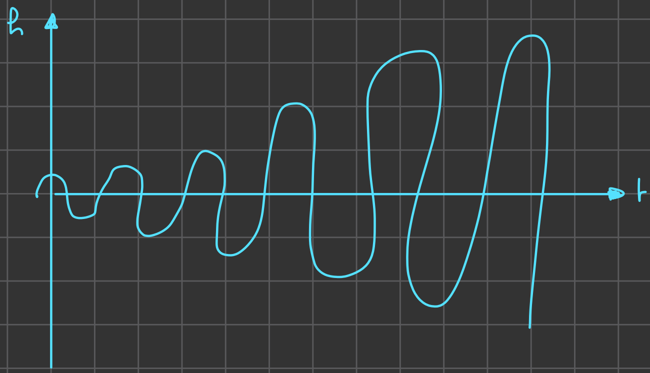
$$= \Phi_0 - 2\left(\frac{5G\mathcal{M}_c}{c^3}\right)^{-5/8} \tau^{5/8}$$

$t \to \tau$
$dt \to -d\tau$

w/ $\Phi_0 \equiv \Phi(\tau = 0) = \Phi(t_{coal})$.

Then, the GW amplitude can be expressed directly in terms of the time to coalescence τ measured by the observer

$$\begin{cases} h_+(\tau) \sim \tau^{-1/4} \cos\left(\tau^{5/8}\right) \\ h_\times(\tau) \sim \tau^{-1/4} \sin\left(\tau^{5/8}\right) \end{cases}$$



(cf Fig. 1)

Problem 2.7

GW150914 detected by LIGO involved 2 inspiralling bodies of masses $36{,}2\, M_\odot$ and $29{,}1\, M_\odot$.

a) Compute the gravitational wave frequency f_{GW} $0{,}02\,s$ before the coalescence. What would R be? What can be said about the nature of this 2-body system?

We know that

$m_1 = 36{,}2\, M_\odot$ $\qquad$ $\tau = 0{,}02\,s$

$m_2 = 29{,}1\, M_\odot$ $\qquad$ $\mathcal{M}_c = 28\, M_\odot$

$$\Rightarrow f_{GW}(\tau) = \dots = 81{,}5\ Hz$$

Kepler allows us to tell

$$R = \left(\frac{4Gm}{\omega_{GW}^2}\right)^{1/3} = \left(\frac{G\cdot m}{\pi^2 f_{GW}^2}\right)^{1/3} = 405\ km$$

$\rightarrow$ both bodies are small, they can be either BHs or ~~NS~~

too massive (cf Chandra. limit)

b) Compute the energy carried away by the GW emission as the GW frequency increases from 10 to 30 Hz.

We have

$$E = \int dt \cdot P = -\int d\tau \cdot C\, \omega_{GW}^{10/3}$$

$$= -\int d\tau \cdot C'\, f_{GW}^{10/3}$$

$$f_{GW} = \tilde{C}\, \tau^{-3/8} \; ; \; df_{GW} = \bar{C}\, \tau^{-11/8}\, d\tau$$

$$= -\int df \cdot C''\, f_{GW}^{10/3}\, \underbrace{\tau^{11/8}}_{C' f_{GW}^{-11/3}}$$

$$= 3\, \tilde{\tilde{C}}\, f_{GW}^{2/3} \sim \ldots M_\odot c^2 \quad \Rightarrow \text{several solar masses}$$

$$\tilde{\tilde{C}} \to \alpha (G M_c)^{5/3} \frac{1}{G}$$